The uncountable Specker phenomenon and n-slender groups

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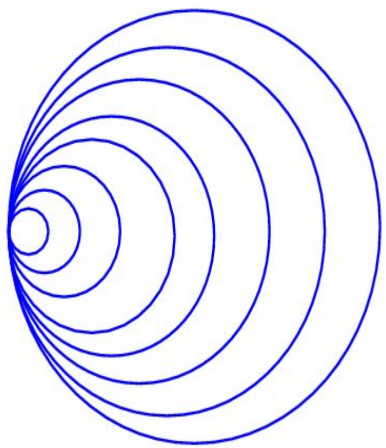

1.Specker'theorem and slender groups

2.Non-commutative Specker'theorem and n-slender groups

3.The uncountable Specker phenomenon

1.Specker'theorem and slender groups

E.Specker(1950)

$h : \mathbb{Z}^\omega \to \mathbb{Z}$ a homomorphism.

$$\exists m \qquad \begin{tikzcd} \mathbb{Z}^\omega \arrow[r, "h"'] \arrow[d, "p_m"'] & \mathbb{Z} \\ \mathbb{Z}^m \arrow[ur, dashed, "\exists \overline{h}"'] & \end{tikzcd}$$

$h = \overline{h} \circ p_m$ $\quad p_m$:projection. $\quad \overline{h}(x) = \sum_{i=0}^{m-1} x(i)h(e_i)$

e_i:i-th component is 1 , other components are all zero.

$$x = \sum_{i<\omega} x(i)e_i = \sum_{i=0}^{m-1} x(i)e_i + \sum_{m\le i<\omega} x(i)e_i$$

$$\begin{aligned} h(x) &= h(\sum_{i=0}^{m-1} x(i)e_i) + h(\sum_{m\le i<\omega} x(i)e_i) \\ &= \sum_{i=0}^{m-1} x(i)h(e_i) + h(\sum_{m\le i<\omega} x(i)e_i) \\ &= \sum_{i=0}^{m-1} x(i)h(e_i) \end{aligned}$$

$h(x)$ is determined by only finite components of x.

h factors through a finitely generated free abelian group $\mathbb{Z}^m$.

Slenderness was introduced by J.Łoś.

An abelian group S is slender, if S satisfies the following diagram.

$h : \mathbb{Z}^\omega \to S$ a homomorphism.

$$\begin{tikzcd}
\mathbb{Z}^\omega \arrow[r, "h"'] \arrow[d, "\exists m \quad p_m"'] & S \\
\mathbb{Z}^m \arrow[ur, dashed, "\exists \overline{h}"'] &
\end{tikzcd}$$

$$h = \overline{h} \circ p_m$$

A slender group S satisfies Specker'theorem.

$\mathbb{Z}$ is a typical example of slender groups.

Theorem (L.Fuchs)

Direct sums of slender groups are slender.

Theorem (R.J.Nunke) the characterization of slender groups.

An abelian group is slender if and only if, it is torsion-free and contains no copy of $\mathbb{Q}, \mathbb{Z}^\omega$, or p-adic integer group $\mathbb{J}_p$ for any prime p.

2.Non-commutative Specker'theorem and n-slender groups

G.Higman (1952)

Let F be a free group and $h : \circledast_{n<\omega}\mathbb{Z}_n \to F$ a homomorphism.

$$\begin{tikzcd} \circledast_{n<\omega}\mathbb{Z}_n \arrow[r, "h"'] \arrow[d, "p_m"', "\exists m"'] & F \\ *_{i<m}\mathbb{Z}_i \arrow[ur, dashed, "\exists \overline{h}"'] & \end{tikzcd}$$

$h = \overline{h} \circ p_m$ p_m: canonical projection

$\circledast_{n<\omega}\mathbb{Z}_n$ is the free complete product of copies of $\mathbb{Z}$.

It is isomorphic to the fundamental group of the Hawaiian earring.

n-slenderness was introduced by K.Eda in 1992.

A group S is n-slender if, G satisfies the following diagram.

$$
\exists m \qquad
\begin{tikzcd}
\circledast_{n<\omega}\mathbb{Z}_n \arrow[r, "h"'] \arrow[d, "p_m"'] & S \\
*_{i<m}\mathbb{Z}_i \arrow[ur, dashed, "\exists\overline{h}"'] &
\end{tikzcd}
\qquad h = \overline{h} \circ p_m
$$

A n-slender group satisfies non-commutative Specker'theorem.

$\mathbb{Z}$ is also a good example of n-slender groups.

Theorem(K.Eda)

Let A be an abelian group.

A is slender if and only if, A is n-slender.

Theorem(K.Eda)

Let $G_i (i \in I)$ be n-slender. Then, the free product $*_{i \in I} G_i$

and the restricted direct product $\prod_{i \in I}^{r} G_i = \{x \in \prod_{i \in I} G_i | \{i \in I | x(i) \neq e\}$ is finite $\}$ are n-slender.

There is a characterization of n-slender groups using fundamental groups.

Theorem(K.Eda)

$\pi_1(X,x)$ is n-slender if and only if,

for any homomorphism $h : \pi_1(\mathbb{H},o) \to \pi_1(X,x)$,

there exists a continuous map $f : (\mathbb{H},o) \to (X,x)$

such that $h = f_*$ where f_* is the induced homomorphism.

We can rephrase Higman's theorem in topological terms as follows:

Let h be a homomorphism from $\pi_1(\mathbb{H}, o)$ to $\pi_1(\mathbb{S}^1)$.

Then, there exists a continuous map $f : \mathbb{H} \to \mathbb{S}^1$ such that $h = f_*$.

Many things about wild algebraic topology can be reduced to the Hawaiian earring and

how the homomorphic image of the fundamental group of the Hawaiian earring can detect a point in the space in question.

It is due to the non-commutative Specker phenomenon.

Theorem(K.Eda)

Let X and Y be a one-dimensional Peano continua which are not semi-locally simply connected at any point.

Then, X and Y are homeomorphic if and only if, the fundamental groups of X and Y are isomorphic.

Theorem(K.Eda)

Let X and Y be one-dimensional Peano continua.

If the fundamental groups of X and Y are isomorphic, then X and Y are homotopy equivalent.

3. The uncountable Specker phenomenon

Theorem 1.(J.Łoś)

Let S be a slender group.

For any homomorphism $h : \mathbb{Z}^I \to S$, there exist ω_1-complete ultrafilters $\mathcal{U}_1, \cdots, \mathcal{U}_n$ on I such that

$h = \overline{h} \circ p_{u_1 \cup \cdots \cup u_n}$ *for any* $u_1 \in \mathcal{U}_1, \cdots, u_n \in \mathcal{U}_n$.

$$
\begin{tikzcd}
\mathbb{Z}^I \arrow[r, "h"'] \arrow[d, "p_{u_1 \cup \cdots \cup u_n}"'] & S \\
\mathbb{Z}^{u_1 \cup \cdots \cup u_n} \arrow[ur, dashed, "\exists \overline{h}"'] &
\end{tikzcd}
$$

Corollary 1.

If κ is less than the least measurable cardinal, then $\mathbb{Z}^\kappa$ satisfies Specker's theorem.

Remark

Let κ be the least cardinal which has a non-pricipal ω_1-complete ultrafilters on κ. Then, κ is measurable.

$$\exists X_0 \Subset \kappa \qquad \begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{Z}^{\kappa} \arrow[r, "h"'] \arrow[d, "p_{X_0}"'] & \mathbb{Z} \\
\mathbb{Z}^{X_0} \arrow[ur, dashed, "\exists \overline{h}"'] &
\end{tikzcd}$$

Theorem 2.(S.Shelah and L.Strüngmann)

$\divideontimes_{\alpha<\omega_1}\mathbb{Z}_\alpha$ *fails the non-commutative Specker phenomenon, i.e;*

there exists a homomorphism $h : \divideontimes_{\alpha<\omega_1}\mathbb{Z}_\alpha \to \mathbb{Z}$ *such that* $h(\delta_\alpha) = 0$ *for any* $\alpha < \omega_1$ *but also* h *is non-trivial.*

Theorem 3.(S.Shelah and K.Eda)

Let S *be a n-slender group.*

For any homomorphism $h : \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I) \to S$*,*

there exist ω_1*-complete ultrafilters* $\mathcal{U}_1, \cdots, \mathcal{U}_n$ *on* I *such that*

$h = \overline{h} \circ p_{u_1\cup\cdots\cup u_n}$ *for any* $u_1 \in \mathcal{U}_1, \cdots, u_n \in \mathcal{U}_n$*.*

$$
\begin{tikzcd}
\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I) \arrow[r, "h"'] \arrow[d, "p_{u_1\cup\cdots\cup u_n}"'] & S \\
\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset u_1 \cup \cdots \cup u_n) \arrow[ur, dashed, "\exists \overline{h}"'] &
\end{tikzcd}
$$

If $|I|$ is less than the least measurable cardinal, then the follwoing holds.

$$\begin{tikzcd}
\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I) \arrow[r, "h"'] \arrow[d, "p_{X_0}"'] & S \\
*_{i\in X}\mathbb{Z}_i \arrow[ur, dashed, "\exists\overline{h}"'] &
\end{tikzcd}$$

$\exists X_0 \Subset I$

If κ is uncountable, $\circledast_{\alpha<\kappa}\mathbb{Z}_\alpha$ fails the non-commutative Specker phenomenon.

But, there exist subgroups of $\circledast_{\alpha<\kappa}\mathbb{Z}_\alpha$ which exhibit the Specker phenomenon.

Def.

Let G_i $(i \in I)$ be groups s.t $G_i \cap G_j = \{e\}$ for any $i \neq j \in I$. we call elements of $\bigcup_{i \in I} G_i$ letters.

A word W is a function

$W : \overline{W} \rightarrow \bigcup_{i \in I} G_i$ $\quad \overline{W}$ is a **linearly ordered set** and

$\{\alpha \in \overline{W} \mid W(\alpha) \in G_i\}$ is **finite** for any $i \in I$.

The class of all words is denoted by $\mathcal{W}(G_i : i \in I)$

Theorem 4.(K.Eda and J.Nakamura)

Let G be the subgroup of $\circledast_{\alpha<\kappa}\mathbb{Z}_\alpha$ consisting of all words which have no subword whose cofinality or coinitiality is uncountable.

Then, G is a maximal subgroup which exhibits the non-commutative Specker phenomenon.

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