高階算術における抽象論

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数学基礎論若手の会　’１０

Contents.

Finite order arithmetic is a formal system based on λ-culiculus.

sorts.

1. $\mathcal{M}_0 \longleftrightarrow \mathbb{N}$
2. $\mathcal{M}_{\sigma\to\tau} \longleftrightarrow$ the set of all maps $\mathcal{M}_\sigma$ to $\mathcal{M}_\tau$

where σ and τ are given sorts.

In short, $0 \to 0$ is denoted by 1. similarly $n \to 0$ is denoted by $n+1$. $\sigma_1 \to (\sigma_2 \to \tau)$ is denoted by $(\sigma_1, \sigma_2) \to \tau$.

Language.

- (Constants) $0, S, \mathcal{R}_0, \cdots,$
- (λ-abstructions) $\lambda x^\sigma.t^\tau$ (the sort is $\sigma \to \tau$.)
- (Applications) $t^{\sigma\to\tau}(s^\sigma)$ (the sort is τ.)

where t and s are given terms, x is a variable symbol.

axiom of λ-caliculus.

- (λ-reduction)

$$(\lambda x^\sigma.t^\tau)(s^\sigma) = t[s/x].$$

- (extentionality)

$$\forall x^{\sigma\to\tau}\forall y^{\sigma\to\tau}(x = y \leftrightarrow \forall z^\sigma(x(z) = y(z))).$$

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There is natural translation from the system of second order arithmetic to finite order arithmetic and finite order arithmetic to set theory.

translation from S.O.A. to F.O.A.

$\mathcal{M}$: A model of finite order arithmetic.
$\longrightarrow (\mathcal{M}_0, \{X \in \mathcal{M}_1 | \forall n \in \mathcal{M}_0 (X(n) \in \{0,1\})\})$.

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V: A model of set theory.

$$\longrightarrow \begin{cases} \mathcal{M}_0 & = \omega^V, \\ \mathcal{M}_{\sigma\to\tau} & = \{f : \mathcal{M}_\sigma \to \mathcal{M}_\tau\}^V. \end{cases}$$

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Strength of S.O.A., F.O.A. and set theory is as follows.

relation of axioms of S.O.A. and F.O.A.

- (Kohlenbach, 2005) An axiom RCA_0^ω of F.O.A., our base axiom, is conservative extension of an axiom RCA_0 of S.O.A.
- (Hunter, 2008) $\mathrm{RCA}_0^\omega + (\mathcal{E}_1)$ is conservative extension of ACA_0.
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- $\mathrm{ZF} \vdash \mathrm{RCA}_0^\omega + \mathcal{E} + \mathrm{Con}(\mathrm{RCA}_0^\omega + (\mathcal{E}))$.
- $\mathrm{ZFC} \vdash \mathrm{RCA}_0^\omega + \mathcal{E} + \mathrm{FAC} + \mathrm{Con}(\mathrm{RCA}_0^\omega + (\mathcal{E}) + \mathrm{FAC})$.

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How do we consider ”abstruct theories” in finite order aritmetic

- Abstract mathematics are formalized by the following sense: If we do not fix the sort, the mood of arbitrary set could be represented.
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2. Definition of axioms of finite order arithmetic.

RCA_0^ω is the axiom consists of the following formulas.

- The axiom of λ-calculus.
- $\forall x^0(\exists y^0(x = S(y)) \leftrightarrow x \neq 0), \forall x^0 \forall y^0(S(x) = S(y) \to x = y)$
- (Existence of primitive recursion operator.)
$$\exists \mathcal{R}_0 \forall f^1 \forall n^0 \forall m^0 \left[\begin{array}{rcl} \mathcal{R}_0(f,n)(0) & = & n, \\ \mathcal{R}_0(f,n)(S(m)) & = & f(m, \mathcal{R}_0(f,n)(m)). \end{array} \right]$$
- (Induction axiom.)
$$\forall A^1(0 \in A \wedge \forall n^0(n \in A \to S(n) \in A) \to \forall n(n \in A)).$$
- (Axiom of choice for $(1,0)$.)
$$\forall A^{(1,0)\to 0}[(\forall x^1 \exists y^0(x,y) \in A) \to (\exists F^{1\to 0} \forall x(x, F(x)) \in A)].$$

Where 0^0 and S^1 are constant symbols.

Definition.

- Q^σ-CA^τ: $\exists X^{\tau\to 0}\forall x^\tau(x \in X \leftrightarrow \varphi(x))$
  where φ is described by $=_0$, Boolean connections and σ variable quantifiers $\exists y^\sigma, \forall y^\sigma$.
- $\mathcal{E}_{\sigma\to 0}$: $\exists E^{(\sigma\to 0)\to 0}\forall x^{\sigma\to 0}(x \in E \leftrightarrow \forall y^\sigma x(y) = 0)$
- $\mathrm{FAC}^{\sigma,\tau}$: $\forall A^{(\sigma,\tau)\to 0}(\forall x^\sigma \exists y^\tau((x,y) \in A) \to \exists F^{\sigma\to\tau}((x, F(x)) \in A))$.
- $\mathrm{GAC}^{\sigma,\tau}$:
  $$\forall A^{(\sigma,\tau)\to 0}\begin{pmatrix} & \forall x^\sigma \exists y^\tau; (x,y) \in A \\ \to & \exists G^{(\sigma,\tau)\to 0}(G \subset A \land (\forall x \exists ! y; (x,y) \in G) \end{pmatrix}.$$

Proposition.

$\{Q^\sigma\text{-}CA^\tau\}_\tau$ is equivalent to $(\mathcal{E}_{\sigma\to 0})$ over RCA_0^ω.

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3. Higher rank axioms imply lower rank axioms.

Definition. (the rank of sorts)

The rank of sort is defined as follows inductively.

$$\begin{aligned} rank(0) &:= 0 \\ rank(\sigma \to \tau) &:= \max(rank(\sigma)+1, \mathrm{rank}(\tau)) \end{aligned}$$

Intuitively, rank is corresponded to the cardinality of the set of all elements. $rank(0) = 0$ means $\mathcal{M}_0$ is countable, rank $= 1$ means continuum, rank $= 2$ is to have cardinality of power set of continuum...

Lemma.

Assume $rank(\sigma) \leq rank(\sigma')$, then the assertion

$$\exists I^{\sigma \to \sigma'} \exists P^{\sigma' \to \sigma} \forall x^{\sigma} (P(I(x)) = x)$$

is provable in RCA_0^{ω}.

Proposition.

Let $\sigma, \sigma', \tau, \tau'$ be sorts and assume $rank(\sigma) \leq rank(\sigma'), rank(\tau) \leq rank(\tau')$. Then the following statements are provable in RCA_0^{ω}.

1. $(\mathcal{E}_{\sigma' \to 0}) \to (\mathcal{E}_{\sigma \to 0})$.
2. $\mathrm{FAC}^{\sigma', \tau'} \to \mathrm{FAC}^{\sigma, \tau}$.
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4. The hierarchy of comprehension axioms.

Theorem.

Let $n \geq 1$ and T be a "Q^n-definable" set of $Q^{\leq n-1}$-sentences. Then the following holds.

$$\mathrm{RCA}_0^\omega + \mathcal{E}_{n+1} + T \vdash \mathrm{Con}(\mathrm{RCA}_0^\omega + \mathcal{E}_n + T).$$

Thus, $\mathrm{RCA}_0^\omega + \mathcal{E}_n$ does not imply $\mathcal{E}_{n+1}$.

The idea of proof: Fix a model $\mathcal{M}$ of $\mathrm{RCA}_0^\omega + \mathcal{E}_{n+1} + T$.
The theorem is proved by some construction in $\mathcal{M}$. It is consists of 3 steps.

1. To construct a model consists of all "λ-terms" generated by $\bigcup_{j \leq n-1} \mathcal{M}_j \cup \{S, \mathcal{R}_0, E_n\} \cup \{\text{variable symbols}\}$.
2. To construct the interpretation of rank$\leq n-1$ elements in $\mathcal{M}$.
3. To construct the graph of the truth value function and to check the axioms.

5. Reverse mathematics related to comprehension.

Definition.

Δ^n_1-CA^n:

$$\forall A \forall B \left[\begin{array}{ll} & \forall x^n (\exists y^n ; (x,y) \in A) \leftrightarrow (\forall z^n ; (x,z) \in B)) \\ \rightarrow & \exists X \forall x (x \in X \leftrightarrow \exists y ; (x,y) \in A) \end{array} \right]$$

Definition.

Σ^n_1-IA:

$$\forall A^{0,n} \left[\begin{array}{ll} & \exists x ; (0,x) \in A \wedge \forall k (\exists x ; (k,x) \in A \rightarrow \exists x ; (S(k),x) \in A) \\ \rightarrow & \forall k \exists x ; (k,x) \in A \end{array} \right]$$

Proposition.

The following statements are equivalent over $\mathrm{RCA}_0^\omega + \Delta_1^n\text{-CA}^n + \Sigma_1^n\text{-IA}$.

1. $[\mathrm{Q}^n\text{-CA}^n \,/\, \mathcal{E}_{n+1}]$.
2. There exists [a subgroup $2A$ / a functional A maps $2A$] for all n-type abelian group (represented by graphs).

compare: RCA_0+"every countable abelian group A, there exists $2A$" implies ACA.

Proposition.

The following statement implies $\mathrm{Q}^n\text{-CA}^n$ over $\mathrm{RCA}_0^\omega + \Delta_1^n\text{-CA}^n + \Sigma_1^n\text{-IA}$:
Every n-type non-zero commutative ring has a maximal ideal.

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6. The hierarchy of choice axioms.

1. Introduction.
2. Definition of axioms of finite order arithmetic.
3. Higher rank axioms imply lower rank axioms.
4. The hierarchy of comprehension axioms.
5. Reverse mathematics related to comprehension.
6. The hierarchy of choice axioms.
7. Reverse mathematics related to choice.

Definition.

1. (σ-WWO) $\exists <(<$ is a well-ordering on $\beth_\sigma$.)
2. (σ-GWO) $\exists <, G \begin{pmatrix} < \text{ is a well-ordering on } \beth_\sigma \\ G = \{(X, y) | X \subset \beth_n, y = minX\} \end{pmatrix}$
3. (σ-FWO) $\exists <, F \begin{pmatrix} < \text{ is a well-ordering on } \beth_\sigma \\ \forall X \neq \emptyset; F(X) = minX \end{pmatrix}$
4. (n-TR)
   $\forall A, X, <; WO(X, <) \rightarrow \exists Y \forall x^n, \alpha^n;$
   $(x, \alpha) \in Y \leftrightarrow (\alpha \in X \wedge (x, \{(y, \beta) | \beta < \alpha, (y, \beta) \in Y\}) \in A)$

Here $WO(X, <)$ is the assertion that $<$ is a well ordering on a set X.

$\{(y, \beta) | \beta < \alpha, (y, \beta) \in Y\}) \in A$ is described as the following form:

$$\exists Z((x, Z) \in A \wedge \forall y, \beta;$$
$$(y, \beta) \in Z \leftrightarrow (\beta \in X \wedge \beta < \alpha \wedge (y, \{(z, \gamma) | \gamma < \beta, (z, \gamma) \in Z\}) \in Z))$$

Thus $\mathrm{RCA}_0^\omega + \mathcal{E}_{n+2} \vdash n\text{-TR}$ holds.

Proposition.

The following statements hold.

1. $\mathrm{RCA}_0^\omega + (\mathcal{E}_{n+2}) + \mathrm{FAC}^{n+1,n} \vdash n\text{-FWO}$.
2. $\mathrm{RCA}_0^\omega + n\text{-FWO} \vdash \mathrm{FAC}^{n+1,n} \wedge (\mathcal{E}_{n+1})$.
3. $\mathrm{RCA}_0^\omega + (\mathcal{E}_{n+2}) + \mathrm{GAC}^{n+1,n} \vdash n\text{-GWO}$.
4. $\mathrm{RCA}_0^\omega + (\mathcal{E}_{n+1}) + n\text{-GWO} \vdash \mathrm{GAC}^{n+1,n}$.

Theorem.

The following statements hold.

$$\mathrm{RCA}_0^\omega + (\mathcal{E}_{n+2}) + \mathrm{FAC}^{n+1,n} \vdash \mathrm{Con}(\mathrm{RCA}_0^\omega + n\text{-FWO}).$$

The proof is the same except $\mathcal{N}$ is generated by $\mathcal{M}_0 \cup \{S, \mathcal{R}_0, <, min\}$ where $<$ is a well ordering of $\mathcal{M}_n$ and min is the map $A \subset \mathcal{M}_n$ to the minimum element of $(A, <)$.

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Definition. (Multiple choice)

$\mathrm{MC}^{\sigma,\tau}$: $\forall A^{(\sigma,\tau)\to 0}$

$$\left[\begin{array}{l} \forall x \exists y((x,y) \in A) \\ \to \exists F^{(\sigma,\tau)\to 0} \left(\begin{array}{ll} & F \subset A \\ \wedge & \forall x \exists y((x,y) \in F) \\ \wedge & \forall x \exists t^0 \exists z^{0\to\tau} \forall y^\tau((x,y) \in F \leftrightarrow \exists s < t(y = z(s))) \end{array}\right) \end{array}\right]$$

Proposition.

$\mathrm{RCA}_0^\omega + \mathcal{E}_{n+2} + \mathrm{MC}^{n+1,n} \vdash n$-GWO.
Especially, $\mathrm{MC}^{n+1,n}$, $\mathrm{GAC}^{n+1,n}$ and n-GWO are equivalent over $\mathrm{RCA}_0^\omega + \mathcal{E}_{n+2}$.

7. Reverse mathematics related to choice.

Proposition.

The following statement is provable in $\mathrm{RCA}_0^\omega + n\text{-FWO} + n\text{-TR}$:
Every n-type vector space over a n-type field has a basis.

Proposition.

"Every n-type vector space over a n-type field has a basis" implies $\mathrm{MC}^{n,n}$ over RCA_0^ω.

compare: The following statements are equivalent over ZF:

1. Axiom of choice in set theory.
2. Multiple choice in set theory.
3. Every vector field has a basis.

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